Klein foams

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A *Klein foam* Ω is a triple (S, Δ, φ), where

- $S = S(\Omega)$ is a family of closed Klein surfaces with boundaries, thus $L = \partial S$ is finite disconnected union of simple contours;
- $\Delta = \Delta(\Omega)$ is a generalized graph: an one-dimensional space, which consists of finitely many vertices and edges, where edges are either segments connecting **different** vertices or isolated circles without vertices on them. A pair of vertices may be connected by several edges;
- $\varphi = \varphi_\Omega : L \to \Delta$ is *the gluing map*, that is, a map such that:
  (a) $\operatorname{Im} \varphi = \Delta$;
  (b) on each connected component of L, φ is a homeomorphism on a circle in Δ;
  (c) for an edge l of Δ any connected component of S contains at most one connected component of $\varphi^{-1}(l \setminus \partial l)$;
  (d) (the normality condition) for $\check{\Omega} = S \cup_\varphi \Delta$ (the result of the gluing of S along Δ) and for each vertex v from the set Ω_b of vertices of the graph Δ, its punctured neighbourhood in $\check{\Omega}$ is connected;
- there exists a continuous function $f : \check{\Omega} \to D$ onto disc $D = \{|z| \leq 1 | z \in \mathbb{C}\}$, that dianalytic and is not constant on any connected component of S.

In classical physics, a particle is a point and its trajectory is a line (world line). In string theory, a particle is an one-dimension object Γ, and a trajectory is a surface S (world surface).

If Γ is a closed contour, then S is a Riemann surface. If Γ is a segment, then S is a Klein surface. If Γ is an arbitrary graph, then S is a Klein foam. Moreover, the string theory requires integration by moduli spaces of the world surfaces. Thus, for the string theory needed topological structures of the moduli spaces.

The moduli spaces of Riemann surfaces consist of connected components homeomorphic to R^m/Mod, where Mod is a discret group (F.Fricke, F.Klein 1897). The moduli spaces of Klein surfaces also consist of connected components homeomorphic to R^m/Mod, where Mod is a discret group (S.Nat. 1975).

Theorem

(A.F.Costa, S.M.Gusen-Zade, S.Nat. 2011) The moduli spaces of Klein foams consist of connected components homeomorphic to R^m/Mod, where Mod is a discret group.

The category of compact Riemann surfaces is isomorphic to the category of complex algebraic curves. The category of compact Klein surfaces is isomorphic to the category of real algebraic curves. That is category of pairs (C,τ), where C is a compact Riemann surface, and $\tau : C \to C$ is an antiholomorphic involution. In this case we say also that τ is a real form of complex algebraic curve corresponding to C.

What is algebra-geometric equivalent of Klein foam?

Let us define an *equipped family of real forms* of a Riemann surface C as a collection $\{C, G, (G_1,\tau_1), \ldots, (G_r,\tau_r)\}$ consisting of:

- a compact Riemann surface C;
- antiholomorphic involutions $\{\tau_1, \ldots, \tau_r\}$ of C;
- subgroups $G_i \subset \mathtt{Aut}(C)$ such that $\tau_i G_i \tau_i = G_i$ for $i = 1, 2, \ldots, r$
- the group G is generated by all G_i and satisfies $\tau_i G \tau_i = G$ for all i.

Theorem

(S.M.Gusen-Zade, S.Nat. 2017) The category of Klein foams is isomorphic to the category of equipped families of real forms of complex algebraic curves.