QCD Pomeron with conformal spin from AdS/CFT Quantum Spectral Curve

Based on
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Motivation

- Using the methods of the recently proposed Quantum Spectral Curve (QSC) originating from integrability of $\mathcal{N} = 4$ Super–Yang-Mills theory analytically continue the scaling dimensions of twist-2 operators and reproduce the so-called pomeron eigenvalue of the Balitsky-Fadin-Kuraev-Lipatov (BFKL) equation with nonzero conformal spin.

- Derive the Faddeev-Korchemsky Baxter equation with nonzero conformal spin for the Lipatov's spin chain known from the integrability of the gauge theory in the BFKL limit.

- Find a way for systematic expansion in the scaling parameter in the BFKL regime and study the Pomeron trajectory by numerical and analytical algorithms of QSC.

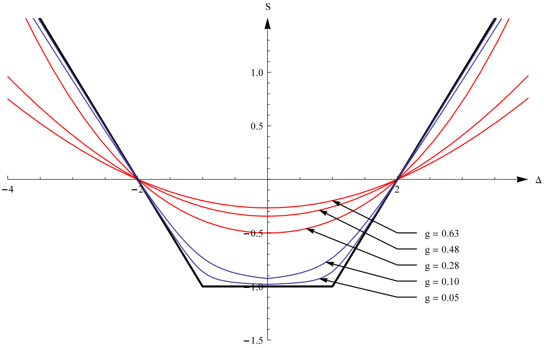
BFKL regime and twist-2 operators in the $\mathcal{N} = 4$ SYM

- We consider important class of operators

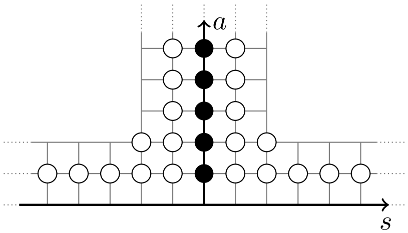
$$\mathrm{tr} Z (D_+)^S (\partial_\perp)^n Z + \text{permutations}$$

For the case with nonzero conformal spin there are derivatives in the orthogonal directions.

- BFKL scaling is determined by: $S \to -1$, $g \to 0$ and $\frac{g^2}{S+1}$ is finite. Leading order BFKL approximation corresponds to resumming all the powers $\left(\frac{g^2}{S+1}\right)^n$.
- Trajectories $S(\Delta)$ corresponding to the twist-2 operator tr $Z(D_+)^S Z$ and different values of g (Gromov, Levkovich-Maslyuk, Sizov'15)



Spectral problem of the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory



- Y-system

$$Y_{a,s}(u + i/2)Y_{a,s}(u + i/2) = \frac{(1 + Y_{a,s+1}(u))(1 + Y_{a,s-1}(u))}{(1 + 1/Y_{a+1,s}(u))(1 + 1/Y_{a-1,s}(u))}.$$

- T-functions

$$Y_{a,s} = \frac{T_{a,s+1}T_{a,s-1}}{T_{a+1,s}T_{a-1,s}}.$$

- Hirota equations

$$T^+_{a,s}T^-_{a,s} = T_{a,s+1}T_{a,s-1} + T_{a+1,s}T_{a-1,s}.$$

Generalities of the QSC

- The QSC gives the generalization of the Baxter equation describing the 1-loop spectrum of twist-2 operators to all loops. The spectrum of the $\mathcal{N} = 4$ SYM can be described by 16 basic Q-functions, which we denote by $\mathbf{P}_a$, $\mathbf{P}^a$, $\mathbf{Q}_j$ and $\mathbf{Q}^j$, where $a, j = 1, \ldots, 4$. (Gromov, Kazakov, Leurent, Volin'13; Gromov, Kazakov, Leurent, Volin'14)

- The AdS/CFT Q-system is formed by 2^8 Q-functions which we denote as $Q_{A|J}(u)$ where $A, J \subset \{1, 2, 3, 4\}$ are two ordered subsets of indices. They satisfy the QQ-relations

$$
\begin{aligned}
Q_{A|I} Q_{Aab|I} &= Q^+_{Aa|I} Q^-_{Ab|I} - Q^-_{Aa|I} Q^+_{Ab|I}\,, \\
Q_{A|I} Q_{A|Iij} &= Q^+_{A|Ii} Q^-_{A|Ij} - Q^-_{A|Ii} Q^+_{A|Ij}\,, \\
Q_{Aa|I} Q_{A|Ii} &= Q^+_{Aa|Ii} Q^-_{A|I} - Q^+_{A|I} Q^-_{Aa|Ii}.
\end{aligned}
$$

  In addition we also impose the constraints $Q_{\emptyset|\emptyset} = Q_{1234|1234} = 1$.

- The poles of Q-functions resolve into cuts $[-2g, 2g]$ (where $g = \sqrt{\lambda}/4\pi$). We have to introduce new objects – the monodromies μ_{ab} and ω_{ij} corresponding to the analytic continuation of the functions $\mathbf{P}_a$ and $\mathbf{Q}_j$ under these cuts.

Generalities of the QSC

- As a consequence of the $\mathrm{Q}\mathrm{Q}$-relations, $\mathbf{P}$'s and $\mathbf{Q}$'s are related through the following 4th order finite-difference equation

$$0 = \mathbf{Q}_j^{[+4]} \mathrm{D}_0 - \mathbf{Q}_j^{[+2]} \left[\mathrm{D}_1 - \mathbf{P}_a^{[+2]} \mathbf{P}^{a[+4]} \mathrm{D}_0 \right] + \frac{1}{2} \mathbf{Q}_j \left[\mathrm{D}_2 + \mathbf{P}_a \mathbf{P}^{a[+4]} \mathrm{D}_0 + \mathbf{P}_a \mathbf{P}^{a[+2]} \mathrm{D}_1 \right] + \text{c.c.}$$

where

$$\mathrm{D}_0 = \det \begin{pmatrix} \mathbf{P}^{1[+2]} & \dots & \mathbf{P}^{4[+2]} \\ \mathbf{P}^{1} & \dots & \mathbf{P}^{4} \\ \mathbf{P}^{1[-2]} & \dots & \mathbf{P}^{4[-2]} \\ \mathbf{P}^{1[-4]} & \dots & \mathbf{P}^{4[-4]} \end{pmatrix}, \quad \mathrm{D}_1 = \det \begin{pmatrix} \mathbf{P}^{1[+4]} & \dots & \mathbf{P}^{4[+4]} \\ \mathbf{P}^{1} & \dots & \mathbf{P}^{4} \\ \mathbf{P}^{1[-2]} & \dots & \mathbf{P}^{4[-2]} \\ \mathbf{P}^{1[-4]} & \dots & \mathbf{P}^{4[-4]} \end{pmatrix},$$

$$\mathrm{D}_2 = \det \begin{pmatrix} \mathbf{P}^{1[+4]} & \dots & \mathbf{P}^{4[+4]} \\ \mathbf{P}^{1[+2]} & \dots & \mathbf{P}^{4[+2]} \\ \mathbf{P}^{1[-2]} & \dots & \mathbf{P}^{4[-2]} \\ \mathbf{P}^{1[-4]} & \dots & \mathbf{P}^{4[-4]} \end{pmatrix}.$$

The four solutions of this equation give four functions $\mathbf{Q}_j$.

$\mathbf{P}\mu$-system

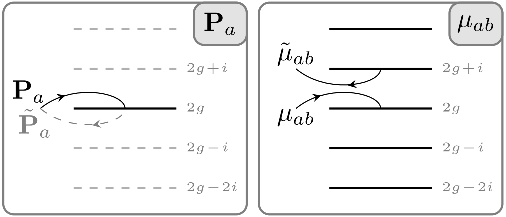
- We can focus on a much smaller closed subsystem constituted of 8 functions $\mathbf{P}_a$ and $\mathbf{P}^a$, having only one short cut on the real axis on their defining sheet

$$\tilde{\mathbf{P}}_a = \mu_{ab}(u)\mathbf{P}^b \ , \ \tilde{\mathbf{P}}^a = \mu^{ab}(u)\mathbf{P}_b,$$

and $\mathbf{P}$'s satisfy the orthogonality relations $\mathbf{P}_a\mathbf{P}^a = 0$.

- The analytic continuation for the μ-functions is given by

$$\tilde{\mu}_{ab}(u) = \mu_{ab}(u+i).$$



- The other equations make the $\mathbf{P}\mu$-system closed

$$\tilde{\mu}_{ab} - \mu_{ab} = \mathbf{P}_a\tilde{\mathbf{P}}_b - \mathbf{P}_b\tilde{\mathbf{P}}_a \ .$$

$\mathbf{Q}\omega$-system

- Knowing $\mathbf{P}_a$ and $\mathbf{Q}_i$ we construct $Q_{a|i}$ which allows us to define ω_{ij}

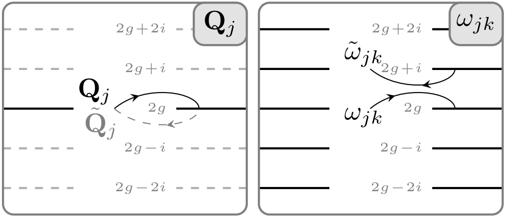
$$\omega_{ij} = Q^-_{a|i} Q^-_{b|j} \mu^{ab}\,.$$

- One can show that $\mathbf{Q}_a$ defined in this way will have one long cut. Also $\hat{\omega}_{ij}$, with short cuts, happens to be periodic $\hat{\omega}^+_{ij} = \hat{\omega}^-_{ij}$, similarly to its counterpart with long cuts $\check{\mu}_{ab}$! Finally, their discontinuities are given by

$$\tilde{\omega}_{ij} - \omega_{ij} = \mathbf{Q}_i \tilde{\mathbf{Q}}_j - \mathbf{Q}_j \tilde{\mathbf{Q}}_i$$

$$\tilde{\mathbf{Q}}_i = \omega_{ij} \mathbf{Q}^j\,.$$

  and $\mathbf{Q}$'s satisfy the orthogonality relations $\mathbf{Q}_j \mathbf{Q}^j = 0$.



Asymptotics of $\mathbf{P}$ and $\mathbf{Q}$-functions and their relation to global S^5 and AdS_5 charges

- The $\mathbf{P}$-functions have the following form on their defining sheet with one short cut

$$\mathbf{P}_a(u) = x^{-\tilde{M}_a}\left(g^{-\tilde{M}_a}A_a + \sum_{k=1}^{+\infty}\frac{c_{a,k}}{x^{2k}(u)}\right),$$
$$\mathbf{P}^a(u) = x^{\tilde{M}_a-1}\left(g^{\tilde{M}_a-1}A^a + \sum_{k=1}^{+\infty}\frac{c^a_k}{x^{2k}(u)}\right).$$

$$\begin{pmatrix}\mathbf{P}_1\\ \mathbf{P}_2\\ \mathbf{P}_3\\ \mathbf{P}_4\end{pmatrix} \simeq \begin{pmatrix} A_1\, u^{\frac{-J_1-J_2+J_3-2}{2}}\\ A_2\, u^{\frac{-J_1+J_2-J_3}{2}}\\ A_3\, u^{\frac{+J_1-J_2-J_3-2}{2}}\\ A_4\, u^{\frac{+J_1+J_2+J_3}{2}}\end{pmatrix} \quad \begin{pmatrix}\mathbf{P}^1\\ \mathbf{P}^2\\ \mathbf{P}^3\\ \mathbf{P}^4\end{pmatrix} \simeq \begin{pmatrix} A^1\, u^{\frac{+J_1+J_2-J_3}{2}}\\ A^2\, u^{\frac{+J_1-J_2+J_3-2}{2}}\\ A^3\, u^{\frac{-J_1+J_2+J_3}{2}}\\ A^4\, u^{\frac{-J_1-J_2-J_3-2}{2}}\end{pmatrix}$$

$$\begin{pmatrix}\mathbf{Q}_1\\ \mathbf{Q}_2\\ \mathbf{Q}_3\\ \mathbf{Q}_4\end{pmatrix} \simeq \begin{pmatrix} B_1\, u^{\frac{+\Delta-S_1-S_2}{2}}\\ B_2\, u^{\frac{+\Delta+S_1+S_2-2}{2}}\\ B_3\, u^{\frac{-\Delta-S_1+S_2}{2}}\\ B_4\, u^{\frac{-\Delta+S_1-S_2-2}{2}}\end{pmatrix} \quad \begin{pmatrix}\mathbf{Q}^1\\ \mathbf{Q}^2\\ \mathbf{Q}^3\\ \mathbf{Q}^4\end{pmatrix} \simeq \begin{pmatrix} B^1\, u^{\frac{-\Delta+S_1+S_2-2}{2}}\\ B^2\, u^{\frac{-\Delta-S_1-S_2}{2}}\\ B^3\, u^{\frac{+\Delta+S_1-S_2-2}{2}}\\ B^4\, u^{\frac{+\Delta-S_1+S_2}{2}}\end{pmatrix}$$

QSC for twist-2 operators with nonzero conformal spin

- Nonzero conformal spin means that $S_2 = n$. These operators do not belong to the so called left-right symmetric sector anymore. But there is still some symmetry

$$\mathbf{P}^a(n,u) = \chi^{ac}\mathbf{P}_c(-n,u), \qquad \mathbf{Q}^i(n,u) = \chi^{ij}\mathbf{Q}_j(-n,u),$$

- The asymptotics are simplified to

$$\begin{aligned}
\mathbf{P}_a &\simeq (A_1 u^{-2}, A_2 u^{-1}, A_3, A_4 u)_a, \\
\mathbf{Q}_j &\simeq (B_1 u^{\frac{\Delta - n + 1 - w}{2}}, B_2 u^{\frac{\Delta + n - 3 + w}{2}}, B_3 u^{\frac{-\Delta + n + 1 - w}{2}}, B_4 u^{\frac{-\Delta - n - 3 + w}{2}})_j
\end{aligned}$$

and

$$\begin{aligned}
A_1 A^1 &= -\frac{1}{96i}((5-w)^2 - (\Delta+n)^2)((1+w)^2 - (\Delta-n)^2), \\
A_2 A^2 &= \frac{1}{32i}((1-w)^2 - (\Delta-n)^2)((3-w)^2 - (\Delta+n)^2), \\
A_3 A^3 &= -\frac{1}{32i}((1-w)^2 - (\Delta+n)^2)((3-w)^2 - (\Delta-n)^2), \\
A_4 A^4 &= \frac{1}{96i}((5-w)^2 - (\Delta-n)^2)((1+w)^2 - (\Delta+n)^2),
\end{aligned}$$

where $w = S + 1$.

Leading Order and Next-to-leading Order solutions of the $\mathbf{P}\mu$-system with conformal spin

- After some demanding calculations we get the result for the $\mathbf{P}$-functions

$$\mathbf{P}_1 \simeq \frac{1}{u^2} + \frac{2\Lambda w}{u^4},$$

$$\mathbf{P}_2 \simeq \frac{1}{u} + \frac{2\Lambda w}{u^3},$$

$$\mathbf{P}_3 \simeq A_3^{(0)} + A_3^{(1)} w,$$

$$\mathbf{P}_4 \simeq A_4^{(0)} u - \frac{i((\Delta^2-1)^2 - 2(\Delta^2+1)n^2 + n^4)}{96u} +$$

$$+ \left(A_4^{(1)} u + \frac{c_{4,1}^{(2)}}{u\Lambda} - \frac{i((\Delta^2-1)^2 - 2(\Delta^2+1)n^2 + n^4)\Lambda}{48u^3} \right) w.$$

where $\Lambda = \frac{g^2}{w}$ and

$$c_{4,1}^{(2)} = -\frac{i\Lambda}{24}(\Delta^2 + n^2 + 2((\Delta-n)^2 - 1)((\Delta+n)^2-1)\Lambda - 1).$$

Passing to $\mathbf{Q}\omega$-system with nonzero conformal spin

- Substituting the obtained LO $\mathbf{P}$-functions into the 4-th order Baxter equation for $\mathbf{Q}$-functions we get a very nice factorization in the LO.
- Thus, we get the equation for $\mathbf{Q}_1$ and $\mathbf{Q}_3$ in the LO

$$\mathbf{Q}_j \frac{(\Delta - n)^2 - 1 - 8u^2}{4u^2} + \mathbf{Q}_j^{--} + \mathbf{Q}_j^{++} = 0,$$

and for $\mathbf{Q}^2$ and $\mathbf{Q}^4$ in the LO

$$\mathbf{Q}^j \frac{(\Delta + n)^2 - 1 - 8u^2}{4u^2} + \mathbf{Q}^{j--} + \mathbf{Q}^{j++} = 0.$$

- In the NLO the 4-th order Baxter equations also factorize and we obtain the following 2nd order Baxter equations

$$\mathbf{Q}_j \left(\frac{(\Delta - n)^2 - 1 - 8u^2}{4u^2} + w \frac{\left((\Delta - n)^2 - 1\right)\Lambda - u^2}{2u^4} \right) + \\ + \mathbf{Q}_j^{--} \left(1 - \frac{iw/2}{u - i}\right) + \mathbf{Q}_j^{++} \left(1 + \frac{iw/2}{u + i}\right) = 0\,, \qquad j = 1, 3.$$

$$\mathbf{Q}^j \left(\frac{(\Delta + n)^2 - 1 - 8u^2}{4u^2} + w \frac{\left((\Delta + n)^2 - 1\right)\Lambda - u^2}{2u^4} \right) + \\ + \mathbf{Q}^{j--} \left(1 - \frac{iw/2}{u - i}\right) + \mathbf{Q}^{j++} \left(1 + \frac{iw/2}{u + i}\right) = 0\,, \qquad j = 2, 4.$$

Calculation of the LO BFKL eigenvalue

- From the NLO 2nd order Baxter equation for $\mathbf{Q}_1$ and $\mathbf{Q}_3$ one can note the following relation between these functions in the LO and NLO

$$\frac{\mathbf{Q}_j^{(1)}(u)}{\mathbf{Q}_j^{(0)}(u)} = +\frac{iw}{2u} + \mathcal{O}(u^0) \;\;,\;\; j = 1,3\,.$$

The key idea of finding the BFKL dimension is to obtain this ratio independently.

- On the other hand we can use the trick

$$\mathbf{Q}_3 = \frac{\mathbf{Q}_3 - \tilde{\mathbf{Q}}_3}{2\sqrt{u^2-4g^2}}\sqrt{u^2-4g^2} + \frac{\mathbf{Q}_3 + \tilde{\mathbf{Q}}_3}{2} =$$

$$= \left[\frac{\mathbf{Q}_3 - \tilde{\mathbf{Q}}_3}{\sqrt{u^2-4g^2}}\right]\left(-\frac{\Lambda w}{u} - \frac{\Lambda^2 w^2}{u^3} + \ldots\right) + \text{regular},$$

from where we conclude that we need to express $\tilde{\mathbf{Q}}_3(u)$ in the LO in terms of $\mathbf{Q}^2(u)$ and $\mathbf{Q}^4(u)$.

- It can be done with some effort, which requires to find ω-functions in the first nonvanishing order. This calculation gives the result

$$\tilde{\mathbf{Q}}_1(u) = -(-1)^n \frac{(\Delta+n)^2-1}{(\Delta-n)^2-1}\mathbf{Q}^2(-u),$$

$$\tilde{\mathbf{Q}}_3(u) = (-1)^n \frac{(\Delta+n)^2-1}{(\Delta-n)^2-1}\mathbf{Q}^4(-u).$$

Calculation of the LO BFKL eigenvalue

- Combining the previously obtained results, we get

$$\mathbf{Q}_1^{(1)}(u) = -\frac{i\mathbf{Q}_1^{(0)}(0)(\Psi(\Delta+n)+\Psi(\Delta-n))\Lambda w}{u} + \text{regular} + \mathcal{O}(w^2),$$

where

$$\Psi(\Delta) \equiv \psi\left(\frac{1}{2}-\frac{\Delta}{2}\right)+\psi\left(\frac{1}{2}+\frac{\Delta}{2}\right)-2\psi(1)\,.$$

- Thus, comparing two independent results, we obtain the relation

$$-2(\Psi(\Delta+n)+\Psi(\Delta-n))\Lambda = 1\,,$$

- After some calculations, we obtain

$$\frac{1}{4\Lambda} = \frac{1}{2}\left(\Psi(\Delta+n)+\Psi(\Delta-n)\right)+\mathcal{O}(g^2) =$$
$$= -\psi\left(\frac{1+n}{2}-\frac{\Delta}{2}\right)-\psi\left(\frac{1+n}{2}+\frac{\Delta}{2}\right)+2\psi(1)+\mathcal{O}(g^2).$$

NLO BFKL eigenvalue with nonzero conformal spin

- The NLO BFKL eigenvalue

$$\delta(n,\Delta) = -2\Phi\left(n, \frac{1-\Delta}{2}\right) - 2\Phi\left(n, \frac{1+\Delta}{2}\right) - 2\zeta(2)\chi(n,\Delta) + \\ + 6\zeta(3) + \Psi''\left(\frac{1+n-\Delta}{2}\right) + \Psi''\left(\frac{1+n+\Delta}{2}\right)$$

can be rewritten as follows for $n = 0$

$$\delta(0,\Delta) = F_2\left(\frac{1-\Delta}{2}\right) + F_2\left(\frac{1+\Delta}{2}\right),$$

where

$$F_2(x) = -\frac{3}{2}\zeta(3) + \pi^2 \log 2 + \frac{\pi^2}{3} S_1(x-1) + \pi^2 S_{-1}(x-1) + 2S_3(x-1) - 4S_{-2,1}(x-1).$$

- Using these results, we are able to rewrite the NLO BFKL eigenvalue for nonzero conformal spin in the following way

$$\delta(n,\Delta) = \frac{1}{2}(\delta(0,\Delta+n) + \delta(0,\Delta-n)) + R_n\left(\frac{1-\Delta}{2}\right) + R_n\left(\frac{1+\Delta}{2}\right),$$

where

$$R_n(x) = -2\left(S_{-2}\left(x+\frac{n}{2}-1\right) + \frac{\pi^2}{12}\right)\left(S_1\left(x+\frac{n}{2}-1\right) - S_1\left(x-\frac{n}{2}-1\right)\right).$$

Gluing conditions. Analytical structure

- The equation for the UHPA $Q_{a|i}$ functions

$$Q_{a|i}^{+} - Q_{a|i}^{-} = \mathbf{P}_a \mathbf{Q}_i.$$

- Hodge dual functions $Q^{a|i}$ are also UHPA and allow to obtain

$$\mathbf{P}^a = \left(Q^{a|i}\right)^{+} \mathbf{Q}_i,$$
$$\mathbf{Q}^i = \left(Q^{a|i}\right)^{+} \mathbf{P}_a,$$

  which are also UHPA and can be analytically continued to the LHP.

- Using the functions $\mathbf{P}_a$ and $\tilde{\mathbf{Q}}^i$, we produce the LHPA functions $Q_a|^i$

$$Q_a|^{i+} - Q_a|^{i-} = \mathbf{P}_a \tilde{\mathbf{Q}}^i$$

  and Hodge dual LHPA system $Q^a|_i$.

- The function

$$\mathbf{P}^a = -(Q^a|_i)^+ \tilde{\mathbf{Q}}^i$$

  has to coincide with the initial one.

- All this gives us

$$\tilde{\mathbf{Q}}^i = -(Q_a|^i Q^{a|j})^+ \mathbf{Q}_j.$$

  The matrix $Q_a|^i Q^{a|j}$ is shown to be i-periodic and antisymmetric in i and j.

Gluing conditions. Algebraic structure

- Due to the determined conjugation properties and parity of the $\mathbf{P}$-functions, $\bar{\mathbf{Q}}_i(u)$ and $\mathbf{Q}_i(-u)$ are also the solutions to the 4th order Baxter equation. Thus, there exist i-periodic matrices that

$$\mathbf{Q}_i(u) = \Omega_i^j(u)\bar{\mathbf{Q}}_j(u),$$
$$\mathbf{Q}_i(u) = \Theta_i^j(u)\mathbf{Q}_j(-u).$$

- Now we are to formulate the general form of the gluing conditions. We have two matrices connecting the Q functions on the different sheets of the Riemann surface (Gromov, Levkovich-Maslyuk, Sizov'15)

$$\tilde{\mathbf{Q}}^i(u) = M^{ij}(u)\bar{\mathbf{Q}}_j(u),$$
$$\tilde{\mathbf{Q}}^i(u) = L^{ij}(u)\mathbf{Q}_j(-u),$$

  where

$$M^{ij} = -(Q_a|^i Q^{a|k})^+ \Omega_k^j,$$
$$L^{ij} = -(Q_a|^i Q^{a|k})^+ \Theta_k^j.$$

- The matrices M and L satisfy

$$\bar{M}^t(u) = M(u), \quad L^t(-u) = L(u).$$

Gluing conditions. Integer and non-integer spins

- We have the restrictions on the gluing matrix

$$M^{ij}(-u)\bar{Q}_j(-u) = L^{ij}(-u)\bar{Q}_j(u),$$
$$M^{ik}\Omega^j_k = -M^{jk}\Omega^i_k.$$

- For the integer spins S_1 and S_2 we obtain

$$M = \begin{pmatrix} 0 & i\alpha & 0 & 0 \\ -i\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & \pm\frac{i}{\alpha} \\ 0 & 0 & \mp\frac{i}{\alpha} & 0 \end{pmatrix}$$

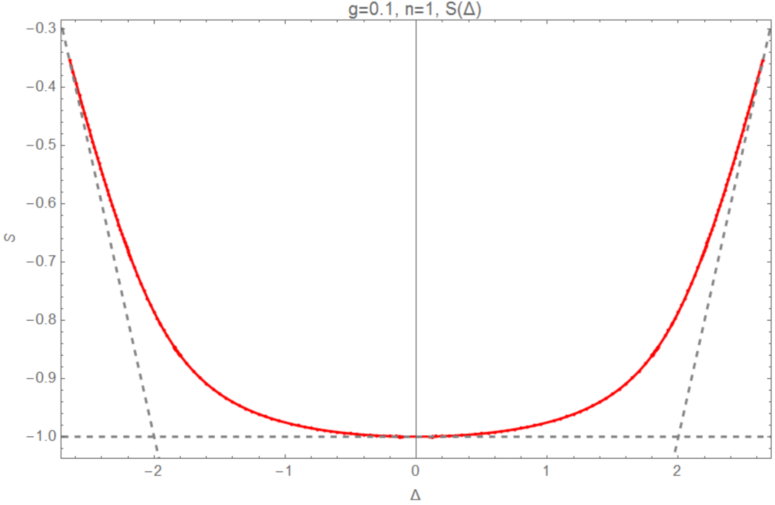
- For non-integer spins S_1 and S_2 we have

$$M = \begin{pmatrix} c_1 & c_2 & c_{31} & c_{41} \\ b_1 & 0 & 0 & 0 \\ d_{31} & 0 & d_1 & d_2 \\ a_{41} & 0 & a_1 & a_2 \end{pmatrix} + \\ + \begin{pmatrix} 0 & 0 & c_{32} & c_{42} \\ 0 & 0 & 0 & 0 \\ d_{32} & 0 & 0 & 0 \\ d_{42} & 0 & 0 & 0 \end{pmatrix} e^{2\pi u} + \begin{pmatrix} 0 & 0 & c_{33} & c_{43} \\ 0 & 0 & 0 & 0 \\ d_{33} & 0 & 0 & 0 \\ d_{43} & 0 & 0 & 0 \end{pmatrix} e^{-2\pi u}.$$

Numerical results for the case $n = 1$

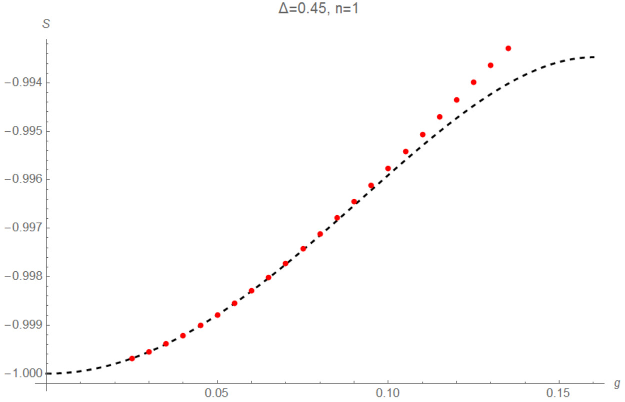
Using the method of Quantum Spectral Curve and asymptotics of the $\mathbf{Q}$-functions described above we are able to numerically calculate (Gromov, Levkovich-Maslyuk, Sizov'15) the following quantities.

- The trajectory $S(\Delta)$ for $g = 1/10$.



Numerical results for the case $n = 1$

- Dependence of S on g for fixed $\Delta = 0.45$.



- Numerical fitting of the BFKL eigenvalues in the first four orders for $\Delta = 0.45$.

	Fit of numerics	Exact perturbative
LO	0.509195398361183370691859	0.509195398361183370691860
NLO	−9.9263626361061612225	−9.9263626361061612225
NNLO	151.9290181554014	?
NNNLO	−2136.77907308	?

BFKL intercept $j(n)$ for general conformal spin n

- Using the binomial harmonic sums

$$\mathbb{S}_{i_1,\ldots,i_k}(M) = (-1)^M \sum_{j=1}^{M} (-1)^j \begin{pmatrix} M \\ j \end{pmatrix} \begin{pmatrix} M+j \\ j \end{pmatrix} S_{i_1,\ldots,i_k}(j)$$

  The known intercept functions in the LO and NLO can be expressed in terms of the binomial harmonic sums with the argument $M = (n-1)/2$

$$j_{\mathrm{LO}} = 4\mathbb{S}_1$$

$$j_{\mathrm{NLO}} = 2(\mathbb{S}_{2,1} + \mathbb{S}_3) + \frac{\pi^2}{3}\mathbb{S}_1,$$

  and allows to formulate an ansatz for NNLO intercept.

- To calculate the intercept the modified iterative procedure from (Gromov, Levkovich-Maslyuk, Sizov'15) was used. But instead of taking different integer values of Δ we used the different values of n. Iterative procedure gives the result for the different values of n, which allows to fix the coefficients of the ansatz. The NNLO intercept is

$$j_{\mathrm{NNLO}} = 32(\mathbb{S}_{1,4} - \mathbb{S}_{3,2} - \mathbb{S}_{1,2,2} - \mathbb{S}_{2,2,1} - 2\mathbb{S}_{2,3}) - \frac{16\pi^2}{3}\mathbb{S}_3 - \frac{32\pi^2}{45}\mathbb{S}_1.$$

  This result is in complete agreement with (Caron-Huot, Herranen'16).

Slope-to-intercept function

- Using the equation
$$Q^+_{a|i} - Q^-_{a|i} = -\mathbf{P}_a \mathbf{P}^b Q^+_{a|i}$$
and knowing that $\mathbf{P}_a$ and $\mathbf{P}^a$ are $\mathcal{O}(n-1)$ in the LO, we find that
$$Q_{a|i} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \mathcal{O}(n-1),$$
- From the weak coupling and numerical data we know that at least ω^{13} and ω^{14} have to be exponential. This allows us to rewrite the equations in the form
$$\begin{aligned}
\tilde{\mathbf{P}}_1 &= \alpha \mathbf{P}_1 + \gamma(\cosh(2\pi u) - I_0)\mathbf{P}_3, \\
\tilde{\mathbf{P}}_2 &= \delta \sinh(2\pi u)\mathbf{P}_1 - \alpha \mathbf{P}_2 + \gamma(\cosh(2\pi u) - I_0)\mathbf{P}_4, \\
\tilde{\mathbf{P}}_3 &= -\alpha \mathbf{P}_3, \\
\tilde{\mathbf{P}}_4 &= -\delta \sinh(2\pi u)\mathbf{P}_3 + \alpha \mathbf{P}_4.
\end{aligned}$$
- Adding the condition that in the asymptotic of μ_{34} there is no logarithmic term in the NLO, we obtain the slope-to-intercept function
$$\theta(g) = 1 + \frac{I_1(4\pi g) I_2(4\pi g)}{\sum_{k=1}^{+\infty} (-1)^k I_k(4\pi g) I_{k+1}(4\pi g)}.$$

Curvature function

- In this case we have the same $Q_{a|i}$ in the LO as in the case of slope-to-intercept function. In the LO we have the following equations

$$\begin{aligned}
\tilde{\mathbf{P}}_1 &= (c_1 + c_3 \cosh 2\pi u)\mathbf{P}^2 + c_4 \mathbf{P}^4 \\
\tilde{\mathbf{P}}_2 &= \frac{\mathbf{P}^3}{c_4} - (c_1 + c_3 \cosh 2\pi u)\mathbf{P}^1 \\
\tilde{\mathbf{P}}_3 &= -\frac{\mathbf{P}^2}{c_4} \\
\tilde{\mathbf{P}}_4 &= -c_4 \mathbf{P}^1 .
\end{aligned}$$

- The curvature function

$$\gamma(g) = \frac{1}{4\pi g^4 I_2^2} \oint_{-2g}^{2g} dv \left(\cosh_-^{v} v \Gamma[\cosh_-^{u} u](v) - \cosh_-^{v} v^2 \Gamma[\cosh_-^{u}](v)\right) +$$

$$+ \frac{1}{16\pi g^5 I_2} \oint_{-2g}^{2g} dv \left(\frac{v^3 \Gamma[\cosh_-^{u}](v) - 2v^2 \Gamma[\cosh_-^{u} u](v) + v\Gamma[\cosh_-^{u} u^2]}{x_v - \frac{1}{x_v}} \right)$$

where

$$\Gamma[h(v)](u) = \oint_{-2g}^{2g} \frac{dv}{2\pi i} \partial_u \log \frac{\Gamma[i(u-v)+1]}{\Gamma[-i(u-v)+1]} h(v).$$

Conclusions and outlook

- In our work we managed to reproduce the dimension of twist-2 operator with conformal spin of $\mathcal{N} = 4$ SYM theory in the 't Hooft limit in the leading order (LO) of the BFKL regime directly from exact equations for the spectrum of local operators called the Quantum Spectral Curve.

- We managed to find two nonperturbative quantities and this is one of a very few examples of all-loop calculations, with all wrapping corrections included, where the integrability result can be checked by direct Feynman graph summation of the original BFKL approach.

- Using the iterative procedure, there was obtained the BFKL intercept for arbitrary conformal spin up to NNLO order in terms of the binomial harmonic sums.

- By application of the QSC numerical algorithm there were calculated the operator trajectories $S(\Delta, n)$ for different values of the conformal spin n and coupling constant g.

- The ultimate goal of the BFKL approximation to QSC would be to find an algorithmic way of generation of any BFKL correction (NNLO (Gromov, Levkovich-Maslyuk, Sizov'15), NNNLO, etc) on Mathematica program, similarly to the one for the weak coupling expansion via QSC.

Thanks for your attention!