One a class of non-local aggregation rules

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Basic model

- A set A of alternatives, $3 \leqslant A < \infty$.
- *Preference functions*, i.e. functions $\mathfrak{c} : [A]^r \to A$ satisfying $\mathfrak{c}(p) \in p$ for all $p \in [A]^r$, where $[A]^r = \{p \subseteq A : |p| = r\}$.
- The set of all preference functions $\mathfrak{C}_r(A)$.
- *Aggregation rules*, i.e. functions $f : (\mathfrak{C}_r(A))^n \to \mathfrak{C}_r(A)$.
- A set of all aggregation rules $\mathcal{F}(A)$

For $r = 2$, each preference function $\mathfrak{c} \in \mathfrak{C}_2(A)$ is associated with the binary *preference relation*

$$P_{\mathfrak{c}} = \{(a, b) \in A^2 : a \neq b \wedge \mathfrak{c}(\{a, b\}) = b\}.$$

The set $\{P_{\mathfrak{c}} : \mathfrak{c} \in \mathfrak{C}_2(A)\}$ is the set of all connex asymmetric binary relations on A.

A function $\mathfrak{c} \in \mathfrak{C}_2(A)$ is called *rational* if $P_{\mathfrak{c}}$ is transitive (cosequently, $P_{\mathfrak{c}}$ is a strict linear order on A). The set of all rational preference functions is denoted $\mathfrak{R}(A)$.

- An aggregation rule $f : (\mathfrak{C}_r(A))^n \to \mathfrak{C}_r(A)$ *preserves* a set $\mathfrak{D} \subseteq \mathfrak{C}_r(A)$ if

$$f(\mathfrak{c}_1, \mathfrak{c}_2, \ldots, \mathfrak{c}_n) \in \mathfrak{D}$$

  for all $\mathfrak{c}_1, \mathfrak{c}_2, \ldots, \mathfrak{c}_n \in \mathfrak{D}$.
- An aggregation rule $f : (\mathfrak{C}_r(A))^n \to \mathfrak{C}_r(A)$ is *local* if
  1. for all $\mathfrak{c}_1, \mathfrak{c}_2, \ldots, \mathfrak{c}_n \in \mathfrak{C}_r(A)$, $p \in [A]^r$ and $a \in p$

$$\mathfrak{c}_1(p) = \mathfrak{c}_2(p) = \ldots = \mathfrak{c}_n(p) = a \Rightarrow f(\mathfrak{c}_1, \mathfrak{c}_2, \ldots, \mathfrak{c}_n)(p) = a;$$

  2. for all $\mathfrak{c}_1, \mathfrak{c}_2, \ldots, \mathfrak{c}_n, \mathfrak{c}'_1, \mathfrak{c}'_2, \ldots, \mathfrak{c}'_n \in \mathfrak{C}_r(A)$ and $p \in [A]^r$

$$(\mathfrak{c}_1(p), \mathfrak{c}_2(p), \ldots, \mathfrak{c}_n(p)) = (\mathfrak{c}'_1(p), \mathfrak{c}'_2(p), \ldots, \mathfrak{c}'_n(p)) \Rightarrow$$
$$\Rightarrow f(\mathfrak{c}_1, \mathfrak{c}_2, \ldots, \mathfrak{c}_n)(p) = f(\mathfrak{c}'_1, \mathfrak{c}'_2, \ldots, \mathfrak{c}'_n)(p).$$

- An aggregation rule $d : (\mathfrak{C}_r(A))^n \to \mathfrak{C}_r(A)$ is *dictatorship* if

$$d(\mathfrak{c}_1, \mathfrak{c}_2, \ldots, \mathfrak{c}_n) = \mathfrak{c}_i$$

  for some $i \in \{1, 2, \ldots, n\}$ and all $\mathfrak{c}_1, \mathfrak{c}_2, \ldots, \mathfrak{c}_n \in \mathfrak{C}_r(A)$.

Impossibility theorems

Theorem (Arrow [1])

There are no local non-dictatorship aggregation rules that preserve the set of rational preference functions (the condition $3 \leqslant |A| < \infty$ is essential).

Theorem (Shelah [2])

If $7 \leqslant r \leqslant |A| - 7$ there are no local non-dictatorship aggregation rules that preserve an arbitrary symmetric non-empty proper subset $\mathfrak{D}$ of $\mathfrak{C}_r(A)$.

Theorem (P., Shamolin [3])

[*Complete classification of symmetric sets of preference functions without the Arrow property.*]

Arrow K. Social Choice and Individual Values. 2 edition. Yale University Press, 1963.

S. Shelah. On the Arrow property. *Advances in Applied Mathematics.* Vol. 34, pp. 217–251, 2005.

Polyakov N., Shamolin M. On a generalization of Arrow's impossibility theorem. *Doklady Mathematics.* Vol. 89, No. 3 , pp. 290–292, 2014.

Non-local aggregation rules

Some non-local aggregation rules partially overcome Arrow's paradox: Borda method, Kemeny–Young method [1], Copeland method [2], Schulze method [3] etc.

The paper [1] proposes a new class of non-local aggregation rules. Key ideas:

- Random factor.
- Simulating of a dynamic aggregation.

Kemeny J. Mathematics without numbers. *Daedalus.* Vol. 88, No. 3 , pp. 577–591, 1959.

Maskin E., Dasgupta P. The Fairest Vote of All. *Scientific American.* Vol. 290, No. 3, pp. 64–69, 2004.

Schulze M. A new monotonic, clone-independent, reversal symmetric, and condorcet-consistent single-winner election method. *Social Choice and Welfare.* Vol. 36, No. 2, pp. 267–303, 2011.

Polyakov N.L., Shamolin M.V. On dynamic aggregation systems. *J. Math. Sci. (N. Y.).* Vol. 244, No. 2, pp. 278–293, 2020.

Definitions

Definition

Let r be a natural number. A r-*lot* (on a set A) is a sequence $(A_0, A_1, \dots, A_k)$ of subsetets of A such that $A_0 = \varnothing$, $|A_1| \geqslant r$, $A_k = A$ and $A_i \subseteq A_{i+1}$ for all i, $1 \leqslant i \leqslant k-1$. An r-*lot* is *maximal* if $|A_1| = r$, and $|A_{i+1} \setminus A_i| = 1$ for all i, $1 \leqslant i \leqslant k-1$.

Definition

Adaptation function is any function

$$\mathcal{A} : \mathfrak{C}_r(A) \times \left(\bigcup_{B \subseteq A} \mathfrak{C}_r(B) \right) \to \mathfrak{C}_r(A),$$

satisfying: for all $B \subseteq A$, $\mathfrak{c} \in \mathfrak{C}_r(A)$ and $\mathfrak{d} \in \mathfrak{C}_r(B)$

1. $\mathcal{A}(\mathfrak{c}, \mathfrak{d}) \restriction_{[B]^r} = \mathfrak{d}$,
2. if $\mathfrak{c} \restriction_{[B]^r} = \mathfrak{d}$ then $\mathcal{A}(\mathfrak{c}, \mathfrak{d}) = \mathfrak{c}$

Adaptation function $\mathcal{A}$ *preserves* the set $\mathfrak{D} \subseteq \mathfrak{C}_r(A)$ if for all $\mathfrak{c} \in \mathfrak{C}(A)$, $B \subseteq A$ и $\mathfrak{d} \in \mathfrak{C}(B)$

$$(\mathfrak{c} \in \mathfrak{D} \wedge \mathfrak{d} \in \mathfrak{D} \restriction_{[B]^r}) \Rightarrow \mathcal{A}(\mathfrak{c}, \mathfrak{d}) \in \mathfrak{D}.$$

For any $B \subseteq A$, each local aggregation function $f : (\mathfrak{C}_r(A))^n \to \mathfrak{C}_r(A)$ can be extended to the set $(\mathfrak{C}_r(B))^n$: for all $\mathfrak{c}'_1, \mathfrak{c}'_2, \dots, \mathfrak{c}'_n \in \mathfrak{C}_r(B)$

$$f(\mathfrak{c}'_1, \mathfrak{c}'_2, \dots, \mathfrak{c}'_n) = f(\mathfrak{c}_1, \mathfrak{c}_2, \dots, \mathfrak{c}_n),$$

where for any i, $1 \leqslant i \leqslant n$, $\mathfrak{c}_i$ is an arbitrary function such that $\mathfrak{c}_i \restriction_{[B]^r} = \mathfrak{c}'_i$.

Definition

For any local n-ary aggregation function f, adaptation function $\mathcal{A}$, lot $J = \{A_0, A_1, \dots, A_m\}$ and profile $(\mathfrak{c}_1, \mathfrak{c}_2, \dots, \mathfrak{c}_n) \in (\mathfrak{C}_r(A))^n$ define the preference function

$$f_{\mathcal{A},J}(\mathfrak{c}_1, \mathfrak{c}_2, \dots, \mathfrak{c}_n)$$

as follow: for any k, $0 \leqslant k \leqslant m$, define preference functions $\mathfrak{c}_1^k, \mathfrak{c}_2^k, \dots, \mathfrak{c}_n^k$ on A and preference function $\mathfrak{d}^k$ on A_k:

1. $\mathfrak{c}_1^0 = \mathfrak{c}_1, \mathfrak{c}_2^0 = \mathfrak{c}_2, \dots, \mathfrak{c}_n^0 = \mathfrak{c}_n$ и $\mathfrak{d}^0 = \varnothing$;
2. if $k \geqslant 1$ then

$$\mathfrak{c}_1^k = \mathcal{A}(\mathfrak{c}_1, \mathfrak{d}^{k-1}), \mathfrak{c}_2^k = \mathcal{A}(\mathfrak{c}_2, \mathfrak{d}^{k-1}), \dots, \mathfrak{c}_n^k = \mathcal{A}(\mathfrak{c}_n, \mathfrak{d}^{k-1})$$

   и

$$\mathfrak{d}^k = f(\mathfrak{c}_1^k \restriction_{[A_k]^r}, \mathfrak{c}_2^k \restriction_{[A_k]^r}, \dots, \mathfrak{c}_n^k \restriction_{[A_k]^r}).$$

Now put

$$f_{\mathcal{A},J}(\mathfrak{c}_1, \mathfrak{c}_2, \dots, \mathfrak{c}_n) = \mathfrak{d}^m.$$

Additional facts

- A set of all aggregation rules preserving a set $\mathfrak{D} \subseteq \mathfrak{C}_r(A)$ is closed w.r.t. composition and contains all dictatorship rules (projections), i.e. it is a *clone* with domain $\mathfrak{C}_r(A)$.
- A clon of all (local) aggregetion rules $f : (\mathfrak{C}_2(A))^n \to \mathfrak{C}_2(A)$, $1 \leqslant n < \infty$, generated by *majority rule* is denoted $\mathcal{M}(A)$. Any n-ary function $f \in \mathcal{M}(A)$ is *neutral*, i.e. for all $a, b, c, d \in A$, $a \neq b$, $c \neq d$, and $\mathfrak{c}_1, \mathfrak{c}_2, \ldots, \mathfrak{c}_n \in \mathfrak{C}_2(A)$ if for all i, $1 \leqslant i \leqslant n$,

$$\mathfrak{c}_i(\{a, b\}) = a \Leftrightarrow c_i(\{c, d\}) = c,$$

  then

$$f(\mathfrak{c}_1, \mathfrak{c}_2, \ldots, \mathfrak{c}_n)(\{a, b\}) = a \Leftrightarrow f(\mathfrak{c}_1, \mathfrak{c}_2, \ldots, \mathfrak{c}_n)(\{c, d\}) = c.$$

- Any local and neutral aggregation rule can by defined by a set $\mathcal{C}_f \subseteq \mathscr{P}(\{1, 2, \ldots, n\})$ of *decisive coalitions*: for all $a \neq b \in A$

$$f(\mathfrak{c}_1, \mathfrak{c}_2, \ldots, \mathfrak{c}_n)(\{a, b\}) = a \Leftrightarrow \{i \in \{1, 2, \ldots, n\} : \mathfrak{c}_i(\{a, b\}) = a\} \in \mathcal{C}_f.$$

- A local and neutral function f belong to $\mathcal{M}(A)$ iff $\mathcal{C}_f$ satisfies:
  1. if $I \in \mathcal{C}_f$ and $I \subseteq J \subseteq \{1, 2, \ldots, n\}$, then $J \in \mathcal{C}_f$,
  2. for any $I \subseteq \{1, 2, \ldots, n\}$ exactly one of the two conditions holds: $I \in \mathcal{C}_f$ and $\{1, 2, \ldots, n\} \setminus I \in \mathcal{C}_f$.

Main results

Further we consider only the case $r = 2$.

Theorem (P., Shamolin [1])

For any set A, $3 \leqslant |A| < \infty$, *local aggregation function* $f : (\mathfrak{C}_2(A))^n \to \mathfrak{C}_2(A)$, *lot* J *and adaptation function* $\mathcal{A}$ *preserving* $\mathfrak{R}(A)$, *the aggregatin function* $f_{\mathcal{A},J}$ *preserves* $\mathfrak{R}(A)$ *iff*

1. J *is maximal,*
2. $f \in \mathcal{M}(A)$.

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1. *J is maximal,*
2. *$f \in \mathcal{M}(A)$.*

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Definition

For any n-ary $f \in \mathcal{M}(A)$ and profile $\boldsymbol{c} = (\mathfrak{c}_1, \mathfrak{c}_2, \ldots, \mathfrak{c}_n) \in (\mathfrak{R}(A))^n$, an element $a \in A$ is called $(f, \boldsymbol{c})$-*winner* if

$$f(\boldsymbol{c})(\{x, a\}) = a$$

for any $x \in A \setminus \{a\}$.

Theorem (P., Shamolin)

For any finite non-empty set A there is an adaptation function $\mathcal{A}_0$ on A such that

1. $\mathcal{A}_0$ *preserves* $\mathfrak{R}(A)$,
2. *for any n-ary $f \in \mathcal{M}(A)$ and profile $\boldsymbol{c} = (\mathfrak{c}_1, \mathfrak{c}_2, \dots, \mathfrak{c}_n) \in (\mathfrak{R}(A))^n$ the $(f, \boldsymbol{c})$-winner $a \in A$ (if it exists) is the maximal element of A w.r.t. linear order $P_{f_{\mathcal{A}_0, J}(\boldsymbol{c})}$ for any maximal lot J.*

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Construction of the function $\mathcal{A}_0$

It suffices to determine the function

$$\mathcal{A}_0' : L(A) \times \left(\bigcup_{B \subseteq A} L(B) \right) \to L(A),$$

satisfying

1. $\prec_2 \subseteq \mathcal{A}_0'(\prec_1, \prec_2)$,
2. if $\prec_2 \subseteq \prec_1$ then $\mathcal{A}_0'(\prec_1, \prec_2) = \prec_1$

for all $(\prec_1, \prec_2) \in \operatorname{dom} \mathcal{A}_0'$.

Definition

For any set $B \subseteq A$, linear order $\prec_B = b_1 b_2 \dots b_k$ on B and linear order $\prec_A$ on A define linear order $\mathcal{A}_0'(\prec_A, \prec_B)$. Let $\prec_{A \setminus B} = a_1 a_2 \dots a_l$ be the restriction of a linear order $\prec_A$ on $B \setminus A$. Define the sequence $\prec_0, \prec_1, \dots, \prec_l$ of linear orders on the sets $B, B \cup \{a_1\}, B \cup \{a_1, a_2\}, \dots, A$ respectively:

1. $\prec_0 = \prec_B$,
2. for all i, $1 \leqslant i \leqslant l$, if $\prec_{i-1} = c_1 c_2 \dots c_{k+i-1}$ then
   - 2.1 if $a_i \prec_A c_r$ for all r, $1 \leqslant r \leqslant k+i-1$ then $\prec_i = a_i c_1 c_2 \dots c_{k+i-1}$,
   - 2.2 if $c_r \prec_A a_i$ for all r, $1 \leqslant r \leqslant k+i-1$ then $\prec_i = c_1 c_2 \dots c_{k+i-1} a_i$,
   - 2.3 otherwise

     $$\prec_i = c_1 c_2 \dots c_j a_i c_{j+1} \dots c_{k+i-1},$$

     where j is the minimal number in $\{1, \dots, k+i-1\}$ for which

     $$a_i \prec_A c_{j+1}, a_i \prec_A c_{j+2}, \dots, a_i \prec_A c_{k+i-1}.$$

Now put $\mathcal{A}_0'(\prec_A, \prec_B) = \prec_l$.

Example. Let $|A| = 3$, $|B| = 2$ and $\prec_A = xyz$.

$\prec_B$	$\mathcal{A}_0(\prec_A, \prec_B)$	$\prec_B$	$\mathcal{A}_0(\prec_A, \prec_B)$
xy	xyz	yx	yxz
yz	xyz	zy	xzy
xz	xyz	zx	zxy

Fact

In general case,

$$f(\boldsymbol{c}) \neq f_{\mathcal{A}_0,J}(\boldsymbol{c})$$

even if $f(\boldsymbol{c}) \in \mathfrak{R}(A)$.

Example. Let $A = \{a, b, c, d\}$, $n = 3$, $f = \mathrm{maj}$, $\mathbf{c} = (\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{c}_3) \in (\mathfrak{R}(A))^3$, $\prec_{\mathfrak{c}_1} = cadb$, $\prec_{\mathfrak{c}_2} = bdac$ и $\prec_{\mathfrak{c}_3} = dabc$. It is easy to check that the preference function $\mathrm{maj}\,(\boldsymbol{c})$ is rational, and

$$\prec_{\mathrm{maj}\,(\boldsymbol{c})} = dabc.$$

Let $J = \{A_0, A_1, A_2, A_3\}$ where $A_0 = \varnothing$, $A_1 = \{b, c\}$, $A_2 = \{a, b, c\}$, $A_3 = \{a, b, c, d\}$. Then we have:

k	A_k	$\mathfrak{c}_1^k$	$\mathfrak{c}_2^k$	$\mathfrak{c}_3^k$	$\mathfrak{c}_1^k \restriction_{[A_k]^2}$	$\mathfrak{c}_2^k \restriction_{[A_k]^2}$	$\mathfrak{c}_3^k \restriction_{[A_k]^2}$	$\mathfrak{d}^k$
0	$\varnothing$	$cadb$	$bdac$	$dabc$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$
1	$\{b, c\}$	$cadb$	$bdac$	$dabc$	cb	bc	bc	bc
2	$\{a, b, c\}$	$bcad$	$bdac$	$dabc$	bca	bac	abc	bac
3	$\{a, b, c, d\}$	$bacd$	$bdac$	$dbac$	$bacd$	$bdac$	$dbac$	$bdac$

$$\prec_{\mathrm{maj}_{\mathcal{A}_0,J}(\boldsymbol{c})} = bdac.$$

Theorem (P., Shamolin)

For any finite non-empty set A, n-ary function $f \in \mathcal{M}(A)$ and profile $\boldsymbol{c} \in (\mathfrak{R}(A))^n$ such that $f(\boldsymbol{c}) \in \mathfrak{R}(A)$ there is a lot J such that $f(\boldsymbol{c}) = f_{\mathcal{A}_0,J}(\boldsymbol{c})$.

Theorem (P., Shamolin)

For any set A, $3 \leqslant |A| < \infty$, n-ary function $f \in \mathcal{M}(A)$ and profile $\boldsymbol{c} = (\mathfrak{c}_1, \mathfrak{c}_2, \ldots, \mathfrak{c}_n) \in (\mathfrak{R}(A))^n$ the following two conditions are equivalent:

1. *For any maximal lot J, $f(\boldsymbol{c}) = f_{\mathcal{A}_0,J}(\boldsymbol{c})$,*
2. *There is a sequence $(a_1, a_2, \ldots, a_{|A|})$ of pairwise distinct elements of A such that for any j, $1 \leqslant j \leqslant |A| - 1$, the set*

$$\{i \in \{1, 2, \ldots, n\} : (a_j, a_{j+1}), (a_j, a_{j+2}), \ldots, (a_j, a_{|A|}) \in \prec_{\mathfrak{c}_i}\}$$

belongs to $\mathcal{C}_f$.

Discution

- Give an axiomatic description of the class of non-local aggregation functions of the form $f_{\mathcal{A},J}$, $f \in \mathcal{M}(A)$.
- Describe all relevant adaptation functions $\mathcal{A}_0$.
- For any n-ary $f \in \mathcal{M}(A)$ and profile $\boldsymbol{c} = (\mathfrak{c}_1, \mathfrak{c}_2, \ldots, \mathfrak{c}_n) \in (\mathfrak{R}(A))^n$, an element $a \in A$ is called $(f, \boldsymbol{c})$-*loser* if for any $x \in A \setminus \{b\}$

$$f(\boldsymbol{c})(\{x, b\}) = x$$

  Is there an adaptation function $\mathcal{A}$ that preserves $\mathfrak{R}(A)$ and satisfies simultaneously the following two conditions: for any n-ary $f \in \mathcal{M}(A)$ and profile $\boldsymbol{c} = (\mathfrak{c}_1, \mathfrak{c}_2, \ldots, \mathfrak{c}_n) \in (\mathfrak{R}(A))^n$

  1. the $(f, \boldsymbol{c})$-winner $a \in A$ (if it exists) is the maximal element of A w.r.t. linear order $P_{f_{\mathcal{A},J}(\boldsymbol{c})}$ for any maximal lot J;
  2. the $(f, \boldsymbol{c})$-loser $a \in A$ (if it exists) is the minimal element of A w.r.t. linear order $P_{f_{\mathcal{A},J}(\boldsymbol{c})}$ for any maximal lot J?
- Provided $f(\boldsymbol{c}) \in \mathfrak{R}(A)$, what other characteristics (besides the maximum element) coincide for the rational preference functions $f(\boldsymbol{c})$ and $f_{\mathcal{A}_0,J}(\boldsymbol{c})$?
- Does the maximum element of order $\prec_{f_{\mathcal{A}_0,J}(\boldsymbol{c})}$ belong to the Smith set, to the Schwartz set?

THANK YOU!